THE FLOW OF A VISCOUS INCOMPRESSIBLE FLUID AT THE ENTRANCE SECTION OF A FLAT CHANNEL

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Results are given for the calculation of velocity and pressure fields for the flow of viscous incompressible fluid in a flat channel when there is a shock velocity profile at the entrance section.

§1. Statement of the problem. Consider the steady flow of a viscous incompressible fluid in a flat gap at the entrance section.

Let the x-axis lie in the direction of flow, and let the y-axis be perpendicular to the planes bounding the flow. The origin lies in the entrance section on one of the plates (Fig. 1). The longitudinal and transverse velocity components and the pressure will be denoted by the symbols u, v, and p, respectively. We now introduce the dimensionless variables

$$x' = \frac{x}{b}, \quad y' = \frac{y}{b}, \quad u' = \frac{u}{u_0},$$

$$v' = \frac{v}{u_0}, \quad \pi = \frac{p}{\rho u_0^2}, \quad R = \frac{u_0 b}{\nu}. \tag{1.1}$$

Here b is the channel half-width, u_0 is some velocity scale, and ρ is the fluid density. The initial equations for the flow of a viscous incompressible fluid in a flat gap then assume the form (the primes are now omitted from the dimensionless variables)

$$u\ \partial u/\partial x + v\ \partial u/\partial y = -\partial\pi/\partial x + R^{-1}\Delta u, \tag{1.2}$$

$$u\ \partial v/\partial x + v\ \partial v/\partial y = -\partial\pi/\partial y + R^{-1}\Delta v, \tag{1.3}$$

$$\partial u/\partial x + \partial v/\partial y = 0. \tag{1.4}$$

For simplicity we consider flow which is symmetric relative to the midplane of the channel, i.e., we take the plane y = 1 as the upper boundary of the flow.

The velocity component u is assumed to have an arbitrary profile at the channel entrance, while the condition for the velocity component is not rigid, namely

$$u = f_1(y), \qquad \partial v/\partial x = 0. \tag{1.5}$$

We now set the following conditions at a distance L from the entrance section:

$$u = f_2(y), \qquad \partial v/\partial x = 0. \tag{1.6}$$

At the channel wall (y = 0) the conditions

$$u = v = 0 \tag{1.7}$$

are naturally satisfied, while at the upper boundary (y = 1)

$$\partial u/\partial y = 0, \qquad v = 0. \tag{1.8}$$

Equations (1.2)–(1.4), together with boundary conditions (1.5)–(1.8), are a closed system which enables us to find the three variables u, v, and π: the pressure π is found to within an arbitrary constant.

§2. Method for solution of the problem. We now transform our system of equations in the following manner. To eliminate the pressure π from it we differentiate Eq. (1.2) with respect to y and Eq. (1.3) with respect to x, and we subtract the first result from the second. We obtain

$$u\ \partial\Omega/\partial x + v\ \partial\Omega/\partial y - R^{-1}\Delta\Omega = 0,$$

$$\Omega = \partial v/\partial x - \partial u/\partial y. \tag{2.1}$$

We now introduce the stream function $\psi(x, y)$, which is an integral of Eq. (1.4)

$$u = -\partial\psi/\partial y, \qquad v = \partial\psi/\partial x. \tag{2.2}$$

Substituting (2.2) into (2.1), we obtain for the required function ψ a fourth-order equation, equivalent to the initial system (1.2)–(1.4)

$$-\frac{1}{R}\Delta(\Delta\psi) - \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y} = 0. \tag{2.3}$$

The boundary conditions for the function ψ in the region under consideration are obtained from the boundary conditions for u and v

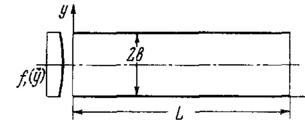

Fig. 1

(1.5)–(1.8) and from relations (2.2):

$$\psi = H_1(y), \qquad \partial^2\psi/\partial x^2 = 0 \quad \text{for} \quad x = 0, \tag{2.4}$$

$$\psi = H_2(y), \qquad \partial^2\psi/\partial x^2 = 0 \quad \text{for} \quad x = L, \tag{2.5}$$

$$\psi = 0, \qquad \partial\psi/\partial y = 0 \quad \text{for} \quad y = 0, \tag{2.6}$$

$$\psi = c, \qquad \partial^2\psi/\partial y^2 = 0 \quad \text{for} \quad y = 1, \tag{2.7}$$

$$H_1(y) = -\int_0^y f_1(y)\,dy, \qquad H_2(y) = -\int_0^y f_2(y)\,dy,$$

$$c = H_1(1) = H_2(1). \tag{2.8}$$

To solve Eq. (2.3) with boundary conditions (2.4)–(2.7) by the method of finite differences, we introduce a mesh consisting of the straight lines

$$x_i = i\ \Delta x \quad (i = 0, 1, \ldots, m+1),$$

$$y_k = k\ \Delta y \quad (k = 0, 1, \ldots, n+1),$$

in the region 0 < x < L, 0 < y < 1, while for simplicity we take $\Delta x = \Delta y = h$ temporarily.

However, before passing to finite differences in Eq. (2.3), we replace this fourth-order equation by the following system of two second-order equations:

$$-\frac{1}{R}\Delta\Omega + u\frac{\partial\Omega}{\partial x} + v\frac{\partial\Omega}{\partial y} = 0, \qquad \Delta\psi = \Omega, \tag{2.9}$$

where u and v are determined from (2.2). The boundary conditions for the function Ω are obtained from Eq. (2.9.2) and conditions (2.4)–(2.7).

We now pass to finite differences in Eqs. (2.9), using the approximations

$$(\Delta\varphi)_{i,k} = (\varphi_{i-1,k} + \varphi_{i+1,k} + \varphi_{i,k-1} + \varphi_{i,k+1} - 4\varphi_{i,k})h^{-2}, \tag{2.10}$$

$$\left(u\frac{\partial\Omega}{\partial x}\right)_{i,k} = \begin{cases} (\Omega_{i,k} - \Omega_{i-1,k})\,|u_{i,k}|/h, & \text{if} \quad u_{i,k} \geqslant 0 \\ (\Omega_{i,k} - \Omega_{i+1,k})\,|u_{i,k}|/h, & \text{if} \quad u_{i,k} < 0 \end{cases},$$

$$\left(v\frac{\partial\Omega}{\partial y}\right)_{i,k} = \begin{cases} (\Omega_{i,k} - \Omega_{i,k-1})\,|v_{i,k}|/h, & \text{if} \quad v_{i,k} \geqslant 0 \\ (\Omega_{i,k} - \Omega_{i,k+1})\,|v_{i,k}|/h, & \text{if} \quad v_{i,k} < 0 \end{cases},$$

$$u_{i,k} = -(\psi_{i,k+1} - \psi_{i,k-1})/2h,$$

$$v_{i,k} = (\psi_{i+1,k} - \psi_{i-1,k})/2h. \tag{2.11}$$

Approximating the derivatives $\partial\Omega/\partial x$ and $\partial\Omega/\partial y$ with unilateral differences (and not with central differences), we aim first of all to obtain a numerical system that is stable for calculation (with a resulting loss of accuracy).

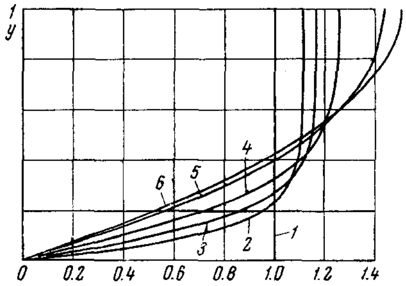

Fig. 2

When we use expressions (2.10) and (2.11), the difference equations for $\Omega_{i,k}$ and $\psi_{i,k}$, the approximating equations (2.9) are written in the following form at the inner points of the mesh:*

$$-\left(\frac{1}{Rh}+\frac{|u_{i,k}|+u_{i,k}}{2}\right)\Omega_{i-1,k}-\left(\frac{1}{Rh}+\frac{|u_{i,k}|-u_{i,k}}{2}\right)\Omega_{i+1,k}-\left(\frac{1}{Rh}+\frac{|v_{i,k}|+v_{i,k}}{2}\right)\Omega_{i,k-1}-\left(\frac{1}{Rh}+\frac{|v_{i,k}|-v_{i,k}}{2}\right)\Omega_{i,k+1}+\left(\frac{4}{Rh}+|u_{i,k}|+|v_{i,k}|\right)\Omega_{i,k}=0, \tag{2.12}$$

$$-\psi_{i-1,k}-\psi_{i+1,k}-\psi_{i,k-1}-\psi_{i,k+1}+4\psi_{i,k}=-h^2\Omega_{i,k}$$
$$(i=1,2,\ldots,m;\ k=1,2,\ldots,n). \tag{2.13}$$

We now write the boundary conditions for the system of functions $\Omega_{i,k}$ and $\psi_{i,k}$. Conditions (2.4) and (2.5) lead to the following conditions at the entrance and exit of the channel:

$$\Omega_{0,k}=G_1(y_k),\qquad \psi_{0,k}=H_1(y_k),$$
$$\Omega_{m+1,k}=G_2(y_k),\qquad \psi_{m+1,k}=H_2(y_k),$$
$$G_1(y)=\partial^2H_1(y)/\partial y^2,\qquad G_2(y)=\partial^2H_2(y)/\partial y^2. \tag{2.14}$$

At the boundary $y = 0$, in accordance with the two conditions (2.6) we have

$$\Omega_{i,0}=(\partial^2\psi/\partial y^2)_{i,0}=(8\psi_{i,1}-\psi_{i,2})/2h^2,\qquad \psi_{i,0}=0.$$

Furthermore, $\psi_{i,1}$ is expressed in terms of $\Omega_{i,1}$ by means of Eq. (2.13). As a result, the conditions for the functions Ω and ψ at the boundary $y = 0$ are written in the form

$$\Omega_{i,0}=-\Omega_{i,1}+(\psi_{i-1,1}+\psi_{i+1,1}+0.5\psi_{i,2})\,h^{-2},\qquad \psi_{i,0}=0. \tag{2.15}$$

At the boundary $y = 1$, in accordance with conditions (2.7),

$$\Omega_{i,n+1}=0,\qquad \psi_{i,n+1}=c. \tag{2.16}$$

After using the boundary conditions (2.14)–(2.16) in Eqs. (2.12) and (2.13), we obtain a system of algebraic equations for the unknowns $\Omega_{i,k}$ and $\psi_{i,k}$ of the form

$$-a_{i,k}\Omega_{i-1,k}-c_{i,k}\Omega_{i+1,k}-b_{i,k}\Omega_{i,k-1}-d_{i,k}\Omega_{i,k+1}+p_{i,k}\Omega_{i,k}=F_{i,k}+g_{i,k}(\psi), \tag{2.17}$$

$$-a'_{i,k}\psi_{i-1,k}-c'_{i,k}\psi_{i+1,k}-b'_{i,k}\psi_{i,k-1}-d'_{i,k}\psi_{i,k+1}+p'_{i,k}\psi_{i,k}=F'_{i,k}-h^2\Omega_{i,k} \tag{2.18}$$
$$(i=1,2,\ldots,m;\quad k=1,2,\ldots,n),$$

where

$$a_{1,k}=c_{m,k}=b_{i,1}=d_{i,n}=0,$$
$$a'_{1,k}=c'_{m,k}=b'_{i,1}=d'_{i,n}=0.$$

On the right-hand side of Eq. (2.17) the term $g_{i,k}(\psi)$, which originates in connection with condition (2.15), is nonzero only on the line $k = 1$, namely

$$g_{i,k}(\psi)=\begin{cases} b^*_{i,1}(\psi_{i-1,1}+\psi_{i+1,1}+0.5\psi_{i,2})\,h^{-2} & \text{for } k=1,\\ 0 & \text{for } k\neq 1,\end{cases}$$
$$b^*_{i,1}=1/Rh+{}^1/_2(|v_{i,k}|+v_{i,k}),$$

if Eq. (2.12) has not been previously multiplied.

We now rewrite the system of equations (2.17)–(2.18) in the following form:

$$-a_{i,k}\Omega_{i-1,k}-c_{i,k}\Omega_{i+1,k}-b_{i,k}\Omega_{i,k-1}-d_{i,k}\Omega_{i,k+1}+p_{i,k}(1+\lambda)\,\Omega_{i,k}=F_{i,k}+g_{i,k}(\psi)+\lambda p_{i,k}(\Delta\psi)_{i,k}, \tag{2.19}$$

$$-a'_{i,k}\psi_{i-1,k}-c'_{i,k}\psi_{i+1,k}-b'_{i,k}\psi_{i,k-1}-d'_{i,k}\psi_{i,k+1}+p'_{i,k}\psi_{i,k}=F'_{i,k}-h^2\Omega_{i,k}. \tag{2.20}$$

The parameter λ appearing in Eq. (2.19) plays the same part qualitatively as the parameter $1/\tau$ (τ is the length of the time step) in the solution of problem (2.9) by another method.

We now split each of Eqs. (2.19) and (2.20) into two first-order difference equations (see [2]), adding the binomial

$$B_{i,k}=r_{i,k}\Omega_{i-1,k+1}+s_{i,k}\Omega_{i+1,k-1}$$

to the right- and left-hand sides of Eq. (2.19). The coefficients $r_{i,k}$ and $s_{i,k}$ of this binomial are as yet undetermined.

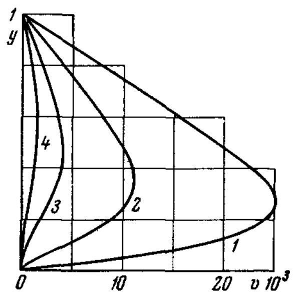

Fig. 3

The system of equations

$$Z_{i,k}=\gamma_{i,k}(a_{i,k}Z_{i-1,k}+b_{i,k}Z_{i,k-1}+F_{i,k}+g_{i,k}(\psi)+\lambda p_{i,k}(\Delta\psi)_{i,k})+\gamma_{i,k}(r_{i,k}\Omega_{i-1,k+1}+s_{i,k}\Omega_{i+1,k-1}), \tag{2.21}$$

$$\Omega_{i,k}=\gamma_{i,k}(c_{i,k}\Omega_{i+1,k}+d_{i,k}\Omega_{i,k+1})+Z_{i,k}, \tag{2.22}$$
$$\gamma_{i,k}=[(1+\lambda)\,p_{i,k}-a_{i,k}c_{i-1,k}\gamma_{i-1,k}-b_{i,k}d_{i,k-1}\gamma_{i,k-1}]^{-1}$$

is equivalent to Eq. (2.19) if the coefficients $r_{i,k}$ and $s_{i,k}$ appearing in the expression for $B_{i,k}$ are taken as equal to

$$r_{i,k}=a_{i,k}d_{i-1,k}\gamma_{i-1,k},\qquad s_{i,k}=b_{i,k}c_{i,k-1}\gamma_{i,k-1}.$$

Similarly, Eq. (2.20) is replaced by the equivalent system

$$W_{i,k}=\gamma'_{i,k}(a'_{i,k}W_{i-1,k}+b'_{i,k}W_{i,k-1}+F'_{i,k}-h^2\Omega_{i,k})+$$

*The transition to finite differences in Eq. (2.9.1) can be performed by more refined methods. The authors also used Allen's method, explained in [1].

$$+\gamma'_{i,k}(r'_{i,k}\psi_{i-1,k+1}+s'_{i,k}\psi_{i+1,k-1}),\tag{2.23}$$

$$\psi_{i,k}=\gamma'_{i,k}(c'_{i,k}\psi_{i+1,k}+d'_{i,k}\psi_{i,k+1})+W_{i,k};$$
$$\gamma'_{i,k}=(p'_{i,k}-a'_{i,k}c'_{i-1,k}\gamma'_{i-1,k}-b'_{i,k}d'_{i,k-1}\gamma'_{i,k-1})^{-1},\tag{2.24}$$
$$r'_{i,k}=a'_{i,k}d'_{i-1,k}\gamma'_{i-1,k},\qquad s'_{i,k}=b'_{i,k}c'_{i,k-1}\gamma'_{i,k-1}.$$

Since the systems of equations (2.21), (2.22) and (2.23), (2.24) are inseparable, namely the coefficients of Eqs. (2.21) and (2.22) and the terms $F_{i,k}+g_{i,k}(\psi)+\lambda p_{i,k}(\Delta\psi)_{i,k}$ depend on the field of the function ψ, we solve both systems of equations together by the method of successive approximations.

Taking the field of the function $\psi_{i,k}$ in the zeroth approximation, we calculate the coefficients $a_{i,k}$, $b_{i,k}$, $c_{i,k}$, $d_{i,k}$, and $\gamma_{i,k}$ and the quantities $F_{i,k}$, $g_{i,k}(\psi)$, $\gamma p_{i,k}(\Delta\psi)_{i,k}$ entering into Eqs. (2.21) and (2.22), and we solve the system of equations (2.21) and (2.22) by iterations. The system of equations (2.23) and (2.24) is then solved (also by iteration). Next, the coefficients $a_{i,k}$, $b_{i,k}$, $c_{i,k}$, $d_{i,k}$, and $\gamma_{i,k}$ of Eqs. (2.21) and (2.22) are determined more accurately, as well as the right-hand side $F_{i,k}+g_{i,k}(\psi)+\lambda p_{i,k}(\Delta\psi)_{i,k}$ of Eq. (2.21), and system (2.21) and (2.22) is solved once more. Then system (2.23) and (2.24) is solved, and so on.

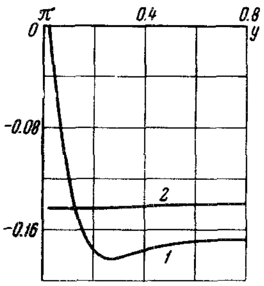

Fig. 4

The iteration process for the whole four-equation system (2.21)–(2.24) is discontinued when the difference between successive approximations for the field of the function $\psi_{i,k}$ becomes less than a given small quantity ε.

We finally obtain the velocity field of u and v in the channel in the form of functions of the coordinates.

The pressure field can be obtained by integration of Eqs. (1.2) and (1.3) directly along the coordinate lines. In this way the pressure is determined at one point of the region under consideration.

§3. Results of the calculations. Problem (1.2)–(1.8) was specifically solved with the following boundary conditions for the velocity components u and v at x = 0 and x = L:

$$u=1,\quad \partial v/\partial x=0\quad \text{at}\quad x=0,\tag{3.1}$$

$$u=3\,y\,(1-{}^1/_2\,y),\quad \partial v/\partial x=0\quad \text{at}\quad x=L.\tag{3.2}$$

Condition (3.2) is taken from the solution of the steady-state problem (1.2)–(1.4) for flow in a channel of infinite extent.

The solution of Eqs. (2.21)–(2.24) was programmed on an electronic computer.

Attempts to solve this system showed, first of all, that for $\lambda = 0$ the iterative process (2.21)–(2.24) can diverge.

The optimum values of λ for which the iterative process (2.21)–(2.24) converges most rapidly are found in the interval $0 < \lambda < 0.5$.

Analysis of the iterative process (2.21)–(2.24) performed enables us to conclude that the iterative terms $r_{i,k}\Omega_{i-1,k+1}+s_{i,k}\Omega_{i+1,k-1}$ on the right-hand side of Eq. (2.21) are very far from being the main part of the right-hand side of this equation. Thus, these terms can be iterated only a few times in the system (2.21) and (2.22). The same can be said of the terms $r'_{i,k}\psi_{i-1,k+1}+s'_{i,k}\psi_{i+1,k-1}$ in Eq. (2.23).

For a number of mesh calculation points $mn \approx 400$ and $\lambda = 0.1$–0.2 the following method of iteration is appropriate: Z and Ω were calculated five times in the process (2.21) and (2.22), and W and ψ were at least once in the process (2.23) and (2.24). Then, to obtain the field of ψ to within three or four figures, the required number of iterations between Eqs. (2.21), (2.22) and (2.23), (2.24) (i.e., the number of external iterations) does not exceed 20–30.

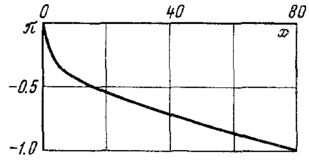

Fig. 5

To obtain a more detailed picture of the fluid flow close to the entrance section and near the channel wall, a mesh with a nonuniform step Δx and Δy was used. This was accomplished by introducing into Eq. (2.9) (before making the transition to finite differences) the new independent variables

$$\mu=\ln\ (1+x/\delta_1),\qquad \nu=\ln\ (1+y/\delta_2),\tag{3.3}$$

in place of the variables x and y, and in which δ_1 and δ_2 are some "scale sizes" of the phenomena close to the boundaries x = 0 and y = 0.

When (3.3) is taken into account, Eqs. (2.9) can be rewritten in the form

$$-\frac{1}{(\delta_1+x)(\delta_2+y)}\frac{\partial\psi}{\partial\nu}\frac{\partial\Omega}{\partial\mu}+\frac{1}{(\delta_1+x)(\delta_2+y)}\frac{\partial\psi}{\partial\mu}\times$$
$$\times\frac{\partial\Omega}{\partial\nu}-\frac{1}{R}\left[\frac{1}{\delta_1+x}\frac{\partial}{\partial\mu}\left(\frac{1}{\delta_1+x}\frac{\partial\Omega}{\partial\mu}\right)+\frac{1}{\delta_2+y}\frac{\partial}{\partial\nu}\times\right.$$
$$\left.\times\left(\frac{1}{\delta_2+y}\frac{\partial\Omega}{\partial\nu}\right)\right]=0,$$
$$\frac{1}{\delta_1+x}\frac{\partial}{\partial\mu}\left(\frac{1}{\delta_1+x}\frac{\partial\psi}{\partial\mu}\right)+\frac{1}{\delta_2+y}\frac{\partial}{\partial\nu}\left(\frac{1}{\delta_2+y}\frac{\partial\psi}{\partial\nu}\right)=\Omega.\tag{3.4}$$

For uniform step lengths of the coordinate mesh in the variables μ and ν, the intervals between calculation points on the linear scale are given by the formulas

$$(\Delta x)_i\approx(\delta_1+x_i)\,\Delta\mu,\qquad (\Delta y)_k\approx(\delta_2+y_k)\,\Delta\nu.$$

Results of the calculations of velocity and pressure fields in the entrance section of a flat gap for various values of the numbers R are given in Figs. 2–6.

As an example, Fig. 2 shows the calculated profiles of the longitudinal velocity component u at various channel sections for R = 500. Curves 1–6 in this figure refer to the sections x = 0, 1.97, 4.05, 10.9, 42.4, and 100, respectively.

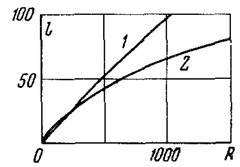

Fig. 6

The calculated profiles of the transverse velocity component v in a flow with the same number R = 500 are shown in Fig. 3. Curves 1–4 refer to the sections x = 1.97, 4.05, 10.9, and 24.8. According to Fig. 3, distance of x ≈ 2 from the entrance the transverse velocity component v reachs a value of 2.5% of the longitudinal velocity component u at the channel entrance.

The calculated pressure π exhibits marked nonuniformity over the channel section only at distances from the entrance within the limits $0 \le x \le 2$. In Fig. 4, curves 1 and 2 describe the pressure distribution at the sections x = 0.987 and x = 2.02 (the value of π is taken as zero at a point with coordinates $x = 0.987$, $y = 0.022$). According to Fig. 4, when $R = 500$ the differential of pressure π at the channel section with $x \approx 1.0$ reaches a value of 0.2 at the wall and inside the flow.

The calculated pressure variation in fluid flow in the direction of the channel is shown in Fig. 5 for a distance of $y = 0.022$ from the channel wall for $R = 500$. It follows from Fig. 5 that for such a value of R the pressure gradient in fluid flow is stabilized at a distance x from the entrance that is on the order of 40.

The calculated hydrodynamic stabilization length l in a laminar fluid flow in a flat gap is presented in Fig. 6 as a function of the number R. Estimates of the length l are made by determining the velocity u at the channel axis (curve 1) and by determining the shearing stress τ at the channel wall (curve 2). Curve 1 is in satisfactory agreement with the results obtained by Leibenzon (see [3]).

The calculated velocity field is in qualitative agreement with the results obtained in [1].

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